9TH CONFERENCE ON COMPETITION AND OWNERSHIP IN LAND TRANSPORT

OPTIMAL PRICING AND FINANCING OF PUBLIC TRANSPORT UNDER COMPETITION

(work in progress♣)

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1 INTRODUCTION

During the last two decades a deregulation trend has flourished. In Western Europe this trend commenced within local and regional public transport. The privatisation of the English bus industry, even the long distance coach services, represents the "full market solution", where both supply, prices and the operation are in the hands of competing profit maximising firms. In the Nordic countries and in London, and to some extent in the US and France, the decision over local and regional public transport supply and prices has been kept in the hands of a public authority, while the actual operation is left for competition through tendering. Typically these services need local or central government grants for financing – and there are economic rationales for this.

Rail transport has in most countries so far been left in the hands of government-controlled bodies, but two exceptions are Great Britain and Sweden. In Great Britain the railway has been split into Railtrack (first public sector organisation, then privatised and then in practice public sector organisation again) and private operators. In Sweden the Swedish Rail (SJ) has been split into a social welfare oriented Railway Administration (Banverket) responsible for infrastructure investments (financed by the government) and the "commercial" new SJ, with the aim to operate the service at a minimum profit determined by the government. SJ still enjoys monopoly for the commercially viable lines for passenger transport, while the non-viable ones are put out for tender by a new government agency (Rikstrafiken).

The aim of this paper is to try to find how one should shape policy intervention in passenger transport markets where there is free competition between operators. One should then first firstly investigate to what extent profit maximising competing operators do not act in accordance with welfare maximisation.

An earlier paper, K. Jansson (2001), examined the discrepancy between profit and welfare

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maximisation with respect to price and frequency and the optimal subsidy in order to make the profit maximising firm behave welfare maximising, but for a single operator without concern for response by competitors. In this earlier work it was shown that optimal price is lower and quality higher under the welfare maximisation scheme compared to the profit maximisation scheme. The main finding was that a subsidy related to the consumer ticket price could yield a welfare optimal solution without any constraint on the policy variables, while a production related subsidy couldn't, except under very restrictive circumstances.

To fix ideas one can have competition between a rail operator and an airline in mind, but the analysis is general for all competing public transport services.

We will first, in section 2, model utility, demand and consumer surplus taking into account that passengers want to minimize generalised cost and that the difference between ideal and actual departure time is an essential decision factor. It appears, unfortunately, that the solutions on optimal price and service frequency are extremely complex and virtually impossible to solve analytically. In section 3 we therefore deviate from the original model and the theoretical findings in section 2 in order to simplify the modelling of competition. Nor the model developed here seems to be solvable analytically. In section 4 we draw some conclusions and discuss what may be the most appropriate steps forward.

A limitation of the work carried out so far is that total demand is kept constant, so that the demand can only shift between the competitors. Given this assumption, numerical tests—not shown in the paper—indicate that profit maximising operators generate a welfare optimum. This might be an interesting result, since we know that a monopolist under fixed demand would set an infinitely high prise and almost zero number of departures, which would be far from welfare optimum. With (Stackelberg) competition between two operators the situation thus seems to change dramatically.

This work is in progress and far from finished. Future work will firstly take into account that total demand is a function of generalised cost of both operators. Secondly, we aim at trying to further employ the original model where demand and generalised cost depends on the difference between ideal and actual departure times of the operators.

2 MODELLING UTILITY, DEMAND AND CONSUMER SURPLUS

In this section we will derive the relations between utility, demand, consumer surplus and choice of mode according to what we think are the most appropriate assumptions about passengers' behaviour. The model is a revised version of Jansson and Lang (2004).

2.1 Basic micro-economic model

We regard passengers travelling in a specific origin-destination (O-D) pair. The aggregate outcome of assignment and consumer surplus is the sum of the outcome of each O-D pair. For each O-D pair passengers are subdivided into segments such that each segment is fairly homogenous with respect to valuation of travel time components in monetary terms. Generalised cost can thus be expressed in monetary terms or minutes. The aggregate outcome of assignment and consumer surplus per O-D pair is simply the sum of the outcome of each segment. We assume that the substitution quotient between time and money is the same for all individuals, i.e. that all have the same valuation of time. We ignore the income effect, which is standard in transport analysis.

Each travel alternative in a specific O-D pair has a generalised cost G. Each individual is assumed to choose the alternative with minimum G. This G is, however, not the same for each individual due to stochastic influence. In order to simplify notation and calculations, without affecting the general aspects, we assume that there are two alternatives, 1 and 2.

The generalised cost of alternative j (j=1,2) for each individual i is composed of the following elements. Travel time R (including all travel time components plus price, except wait time) plus a stochastic variable, t, that varies among individuals with taste, measurement errors etc. plus a stochastic variable, x, that varies among individuals with ideal departure or arrival time in relation to actual time. We define x as time to departure, i.e., the difference between actual and ideal departure time. The generalised cost of alternative j is then:

$$(1) G^j = R^j + t_i^j + x_i^j$$

When each individual chooses the alternative with the minimum generalised cost the realised "joint" generalised cost of individual i is:

$$(2) G_i = \min\left\{R^1 + t_i^1 + x_i^1, R^2 + t_i^2 + x_i^2\right\}$$

The average joint generalised cost of both alternatives over all individuals in a segment is then:

$$(3) G = E\left[\min\left\{R^1 + t_i^1 + x_i^1, R^2 + t_i^2 + x_i^2\right\}\right]$$

where E denotes the expected value corresponding to the distribution of individuals.

We have thus defined one single G for a journey from door to door when there are several alternatives to choose among. The deviation ε_i from the joint G for an individual is defined by:

$$(4) \mathrm{G_i} = \mathrm{G} + \varepsilon_\mathrm{i}$$

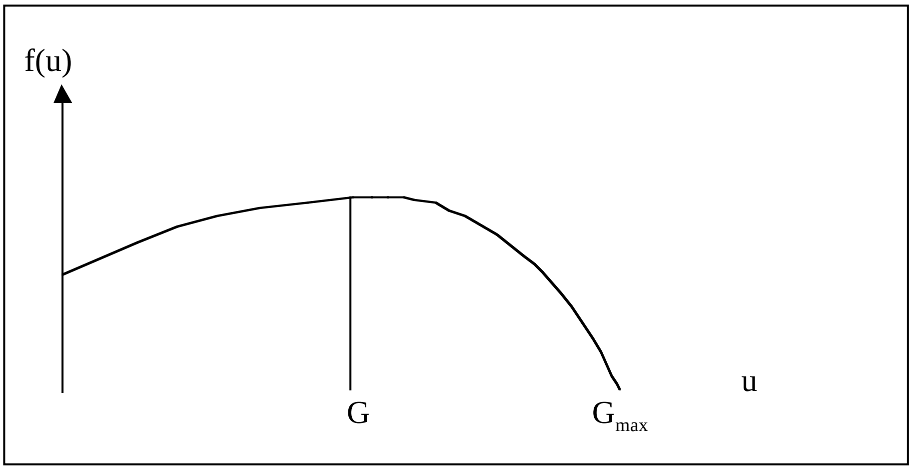
Each individual is assumed to have a utility of travelling from origin to destination, i.e., the utility of the journey itself, which is denoted v_i. The net utility for individual i, when taking G into account, is:

$$(5) \mathrm{v_i} - \mathrm{G}_i = \mathrm{v_i} - \varepsilon_\mathrm{i} - \mathrm{G} \equiv \mathrm{u_i} - \mathrm{G}$$

Let f(u) be the density function over u_i among the individuals.

The individual chooses to travel if $u_i \geq G$, where u_i has a distribution f(u) over all individuals. The choice is illustrated in the diagram below.

Diagram 2.1 Distribution of utility



The aggregate demand, X, is the integral over f(u) between G and the reservation price G_{max}.

$$(6)\ X = \int_{G}^{G_{max}} f(u)\,du = X(G)$$

The consumer surplus, S, is thus:

$$(7)\ S = S(G) = \int_{G}^{G_{max}} (u - G)\,f(u)\,du$$

It then follows that:

$$(8)\ \frac{\partial S}{\partial G} = -(G - G)\,f(G) + \int_{G}^{G_{max}} -f(u)\,du = -X$$

Observe that consumer surplus is a function of the joint generalised cost, which in turn is a function of the generalised cost of both alternatives.

$$(9)\ S = S(G) = S(G(G_1, G_2)) = S(G_1, G_2)$$

Furthermore, note that the stochastic variables x_i, which reflect time to departure of each alternative, depend on the intervals of the alternatives, denoted H^1 and H^2. This means that:

$$(10)\ G = G(R^1, R^2, H^1, H^2)$$

It can be shown, see appendix, that:

$$(11)\ \frac{\partial G}{\partial R^i} = Pr(i)$$

There is, however, no such simple relation between the differential of G with respect to the interval H^i, due to the complex interactions of intervals between the alternatives.

2.2 Consumer surplus

Assume the case where the total demand is constant X, so that we deal with demand variations between alternatives only*. The gross consumer surplus (net of generalised cost) is denoted S*. The consumer surplus for the joint generalised cost G is then:

$$(12)\ S = S^* - X\,G$$

By use of (11) and (8) we know that the derivative of S with respect to a change of travel time of alternative 1 is:

$$(13)\ \frac{\partial S}{\partial R^1} = \frac{\partial S}{\partial G}\frac{\partial G}{\partial R^1} = -X\,Pr(1)$$

* One could of course, without violating the principle analysis in focus here, add another demand model that takes into account that total demand is a function of the joint generalised cost.

The total differential of S is:

$$(14)\, dS = -X(\Pr(1)\,dR^1 + \Pr(2)\,dR^2)$$

A change of R^1 from R^1_1 to R^1_2 yields a change in consumer surplus by the amount:

$$(15)\, \Delta S = X \int_{R_2^1}^{R_1^1} \Pr(1)\,dR^1$$

This means that the change of consumer surplus can be calculated with respect to change of travel time of alternative 1 only, but it does not hold for change of the interval of alternative 1.

Calculation of consumer surplus by taking into account the change of generalised cost of one alternative is thus possible only when the change concerns travel time, i.e., all travel time components plus price, except wait time. If the change concerns the interval of any alternative one must calculate change of consumer surplus by use of change of the joint generalised cost. The reason is that the intervals of the alternatives give rise to the complex interrelationships between alternatives.

2.3 Variation with respect to ideal departure or arrival time

To use the stochastic variable, t that varies among individuals with taste, measurement errors etc. is the principle of for example the logit model.

Our view, however, is that the logit model seems not appropriate for public transport applications. The main explanation is the feature of the model that alternatives are independent. The stochastic variation taken into account in the logit model is assumed to be independent of the measured generalised cost while in fact the generalised cost differs between individuals due to the ignored fact that the generalised cost varies due to variation in ideal departure times. The model can thus not handle the fact that public transport services are “co-operating” via the intervals. More alternatives means that the possibility to get closer to the ideal departure or arrival time increases, an effect that the logit model cannot reflect.

Our conclusion with respect to modelling of demand for public transport modes is to use stochastic variation with respect to differences between actual and ideal departure times. We are thus left with the stochastic element x, difference between actual and ideal departure time, often called schedule delay. This delay is here based on expected delay based on average frequencies of services and not on exact departure times. Expression (3) is then:

$$(16)\, G = E\left[\min\left\{R^1 + x_i^1, R^2 + x_i^2\right\}\right]$$

We define the following indicator functions:

(17)

$$\theta^1 = \begin{cases} 1 & \text{if } 1 \text{ is chosen} \\ 0 & \text{if } 2 \text{ is chosen} \end{cases}$$

$$\theta^2 = \begin{cases} 1 & \text{if } 2 \text{ is chosen} \\ 0 & \text{if } 1 \text{ is chosen} \end{cases}$$

The joint generalised cost is then:

$$(18)\, G = E\left[(R^1 + x^1)\theta^1 + (R^2 + x^2)\theta^2\right] = E\left[R^1\theta^1 + R^2\theta^2\right] + E\left[x^1\theta^1\right] + E\left[x^2\theta^2\right] =$$
$$= R^1 E\left[\theta^1\right] + R^2 E\left[\theta^2\right] + \Pr(1)E\left[x^1|1\right] + \Pr(2)E\left[x^2|2\right] =$$
$$= \Pr(1)R^1 + \Pr(2)R^2 + \Pr(1)E\left[x^1|1\right] + \Pr(2)E\left[x^2|2\right] = T + \Pr(1)E\left[x^1|1\right] + \Pr(2)E\left[x^2|2\right]$$

In (18) E denotes the conditional expectation of x^j conditional on that route number j is chosen. The two last terms define the expected wait time.

The joint generalised cost is thus composed of the average expected travel time (except wait time), above denoted T, and the expected wait time.

We assume now that (x^1, x^2) has a uniform distribution on [0, H^1]×[0, H^2]. This assumption is in turn based on the assumption that we do not now anything about the true distribution about ideal departure times for the period of time (peak hours or non-peak hours for example) we are analysing. We also assume that departure times of alternative routes are uniformly distributed.

The probability of choice of alternative 1 is then:

$$(19)\, \Pr(1) = \frac{1}{H^1 H^2} \int_0^{H^1}\int_0^{H^2} h(R^2 - R^1 + x^2 - x^1)\, dx^2 dx^1$$

where h(s) is the heaviside function defined by:

$$(20)\, h(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s \le 0 \end{cases}$$

RDT thus assumes that passengers know the timetable and choose route, stop and mode, taking into account all travel time components and price and how well ideal departure times relate to actual departure times. It is assumed that departure times are uniformly distributed, with the argument that one normally does not know how departure times between routes are or will be distributed.

The algorithms of the RDT model applied in Sweden are also found in Hasselström (1981) where expression (19) is formulated by use of a single integral, as:

$$(21)\, \Pr(u) = \int_0^{H^u} \frac{1}{H^u} \prod_{\substack{j=1 \\ j\neq u}}^{k} (1 - \min\left\{1, \max\left\{\frac{x^u + R^u - R^j}{H^j}, 0\right\}\right\})\, dx^u$$

Notation

k	number of acceptable routes
H^u	headway of route u
H^j	headway of route j

R^u travel time (including price expressed in minutes) of route u
R^j travel time (including price expressed in minutes) of route j
x^u time to departure of route u

Note that the probability for choice of a specific route depends on travel times, prices and intervals of all acceptable routes.

Note that R^u and R^j may have a different weight in relation to the weight of the headway. If the weight of headway is w the cost of headways are w H^u and w H^j. Hasselström also derives the expected wait time, V, based on the same model as:

$$(22)\, V = \sum_{u=1}^{k} \int_{0}^{H^u} \frac{1}{H^u} \times \prod_{\substack{j=1 \\ j \neq u}}^{k} \left(1 - \min\left\{1, \max\left\{\frac{x^u + R^u - R^j}{H^j}, 0\right\}\right\}\right) dx^u$$

The average expected travel time when there are several acceptable routes is found by the weighted travel time for all routes where the weights are the calculated probabilities. If there are j acceptable routes and the travel time for route i is R^i and the probability of choice of route i is denoted Pr(i), the average expected travel time is:

$$(23)\, T = \sum_{i=1 \rightarrow j} \Pr(i) R^i$$

The generalised cost is simply the sum of the joint expected wait time and the average expected travel time:

$$(24)\, G = V + \sum_{i=1 \rightarrow j} \Pr(i) R^i$$

Note that this procedure for calculation of probabilities and travel time components appears at origin stop and transfer stops. The final generalised cost is composed of the sum of the travel time components and price of each trip-leg.

The model described here may be called RDT, stemming from **R**andom **D**eparture **T**imes. This is a network model implemented in the commercial package Vips and recently also in Visum.

Unfortunately, this “first-best approach” leads to cross-derivatives between the prices and frequencies, and first-order conditions with respect to prices and frequencies that are extremely complex. In addition there appear six different cases dependent on the relations between price plus ride time and frequency of the two alternatives. In the appendix part of these complexities are shown.

For this reason we decided, at least for the time being, to leave the original modelling and simplify the basic analysis. This simplified modelling of competition is provided in section 3 below.

3 MODELLING OF COMPETITION

There are two operators, A and B who compete for the customers. Every customer travels with either of the operators, the choice depending on personal preference or other convenience, price and the frequency offered by the operators. The price p and frequency F is merged into a generalized price G = p+Vw/2F, where Vw is the customer's value of waiting time per hour, such that customers make their choice of operator based on the generalized price, rather than the pecuniary price and frequency separately.

Operators compete according to Stackelberg equilibrium: operator A is the leader and B is the follower. This means the following: For any generalized price G_A of operator A, operator B computes its optimal (i.e., profit maximizing) price $p_B(G_A)$ and frequency $F_B(G_A)$ and hence generalized price $G_B(G_A)$. The leading operator A takes these reaction functions into account when he chooses his optimal price p_A and frequency F_A .The equilibrium is thus described by the following maximization problems, where $\pi_A(p_{A.,} F_{A,} G_B)$ and $\pi_B(p_{B.,} F_{B,} G_A)$ are the profit functions facing operator A and B:

$$(25)\max_{p_B,F_B}\pi_B(p_B,F_B;G_A)\Rightarrow G_B(G_A)$$

$$(26)\max_{p_A,F_A}\pi_A(p_A,F_A;G_B)(p_A+\frac{V_W}{2F_A}))$$

3.1 The Customers

Each customer chooses either of the two operators, taking the generalized prices G_A and G_B into account. Each customer is also endowed with a parameter of preference θ such that his utility of travelling with operator A is u – θ and the utility of travelling with operator B is u + θ - a. Here a is a parameter, and the values of θ are uniformly distributed on the interval $0 \leq \theta \leq a$. There are N customers in total, so for any given θ^* there are N θ^*/a customers whose θ is less than θ^* and N(a– θ)/a whose θ exceeds θ^*.

When the generalized costs are taken into account, the net utility for a customer is u – θ – G_A to travel with operator A, and u + θ – a – G_B to travel with operator B. He chooses to travel with the operator that yields the highest utility; hence the total number of passengers for operator A and B are respectively

$$(27)X_A(G_A,G_B)=\frac{N}{2a}(a+G_B-G_A)$$

$$(28)X_B(G_B,G_A)=\frac{N}{2a}(a+G_A-G_B)$$

3.2 Costs

Each operator n has a cost d_n per passenger, and a cost C_n per departure. Hence the net revenues for operator A and B are, respectively,

$$(29)X_A(G_A,G_B)(p_A-d_A)-C_AF_A=\frac{N}{2a}(a+G_B-G_A)(G_A-\frac{V_W}{2F_A}-d_A)-C_AF_A$$

$$(30)X_B(G_B,G_A)(p_B-d_B)-C_BF_B=\frac{N}{2a}(a+G_A-G_B)(G_B-\frac{V_W}{2F_B}-d_B)-C_BF_B$$

3.3 The Equilibrium

First we derive operator B: s reaction function. For any generalized price G_A of operator A, operator B maximizes his net revenue (we prefer to maximize over G and F rather than p and F; it is of course equivalent:)

$$(31) \max_{G_B, F_B} \frac{N}{2a}(a + G_A - G_B)(G_B - \frac{V_W}{2F_B} - d_B) - C_B F_B$$

In order to ease notation we introduce

$$h = \frac{V_W}{2F_B}$$

$$\tilde{c} = \frac{aC_B V_W}{N}$$

The maximisation problem above is then equivalent to

$$(32) \max_{G_B, h}(a + G_A - G_B)(G_B - h - d_B) - \frac{\tilde{c}}{h}$$

which yields, after some simplification, the two first order conditions:

$$(33) G_B = \frac{1}{2}(a + d_B + G_A + h)$$
$$(34) h^3 - h^2(a - d_B + G_A) + 2\tilde{c} = 0$$

The first equation is always consistent with a maximum; the second equation, which is a third degree equation, may however not. In order for a solution of this equation to represent a minimum, the solution must be

$$(35) h = \frac{y^3}{3} - \frac{2}{3} y \sin(\frac{1}{3} \arcsin z)$$

where

$$y = a - d_B + G_A$$

$$z = 1 - \frac{27\tilde{c}}{y^3}$$

and we must assume that

$$27\tilde{c} < 2y^3$$

After substituting back these results, we get the reaction functions of operator B:

(36)

$$p_B = p_B(G_A)$$
$$F_B = F_B(G_A)$$
$$G_B(G_A) = p_B(G_A) + \frac{V_W}{2F_B(G_A)}$$

We can now turn to the maximization problem of firm A:

$$(37)\max_{G_A,F_A}\frac{N}{2a}(a+G_A-G_B(G_A))(G_B-\frac{V_W}{2F_B}-d_B)-C_BF_B$$

This problem is, however, too complicated to solve analytically, so we employ numerical maximisation.

3.4 Welfare

For any equilibrium configuration we can easily compute the total welfare W. Customers whose θ-value is less than $\theta^* = 0{,}5(a + G_B - G_A)$ will choose operator A. The median of these customers has a θ-value of $0.5\theta^*$ and his net utility is thus $u - 0.5\theta^* - G_A$. Since the θ-values are uniformly distributed, the total consumer surplus of the customers choosing operator A, S_A, is the surplus of the median customer times the number of customers:

$$(38)S_A=(u-0.5\theta^*-G_A)\frac{N\theta^*}{a}$$

and similarly

$$(39)S_B=(u-0.5a+0.5\theta^*-G_B)\frac{N(a-\theta^*)}{a}$$

After simplification we get

$$(40)S=S_A+S_B=Nu-\frac{N}{4a}(a^2+2a(G_A+G_B)-(G_A-G_B)^2)$$

The producer surplus is the sum of the net revenues of the two operators:

$$(41)\pi=\pi_A+\pi_B=$$
$$=\frac{N}{2a}(a+G_B-G_A)(p_A-d_A)-C_AF_A+\frac{N}{2a}(a+G_A-G_B)(p_B-d_B)-C_BF_B$$

And, lastly,

$$(42)W=S+\pi$$

4 Conclusions and future work

Since an analytical solution seemed impossible we carried out some numerical tests. We calculated optimal price, frequency and profit level for both operators. We also calculated the welfare level. It then appeared that insertion of a subsidy or tax on price or frequency would not increase the welfare level. And, by varying prices and frequencies directly in the welfare function did not yield a higher welfare level than what the profit maximising competitors achieved. The optima obtained by the operators were thus found to be also the welfare optimum.

However, a serious limitation of the work carried out so far is that total demand is kept constant, so that the demand can only shift between the competitors. Given this assumption it may not be so astonishing that also welfare is optimal. This might anyway be an interesting result, since we know that a monopolist under this circumstance would set an infinitely high prise and almost zero number of departures, which would be far from welfare optimum. With (Stackelberg) competition between two operators the situation thus seems to change dramatically.

This work is in progress, and the aim is far from finished.

Future work will firstly take into account that total demand is a function of generalised cost of both operators. Secondly, we aim at trying to further employ the original model where demand and generalised cost depends on the difference between ideal and actual departure times of the operators.

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APPENDIX: DERIVATIONS OF RELATIONSHIPS FOR ORIGINAL MODELLING

Here we present our original modelling work based on what we believe is be the best representation of passengers' behaviour, but where the solution turns out to be extremely complex.

Notation

- p is the price for a trip,
- F_i is frequency in number of departures per hour of alternative i,
- H_i is the interval of alternative I
- x_i is the time till departure of alternative i
- t is the ride time,
- c_L is a cost proportionate to number of passengers, mainly sales costs,
- C is the fixed cost per departure,
- X is the total number of passengers for both operators during a period of time.

Policy variables are price, p, and frequency of service, F.

In this appendix sub indices 1 and 2, compared to A and B denote the operators in the main text.

Total demand is a function of price and frequency of both operators, i.e.:

$$(A1) X = X(p_1, p_2, F_1, F_2)$$

The follower maximises:

$$(A2) \pi_2 = p_2 X_2[p_1, p_2\ F_1,\ F_2] - F_2 C_2 - c_{L2} X_2 - f_2 F_2$$

The first-order conditions with respect to price and frequency are:

$$(A3) \frac{\partial \pi_2}{\partial p_2} = X_2 + p_2 \frac{\partial X_2}{\partial p_2} - c_{L2} \frac{\partial X_2}{\partial p_2} = 0$$

$$(A4) \frac{\partial \pi_2}{\partial F_2} = p_2 \frac{\partial X_2}{\partial F_2} - c_{L2} \frac{\partial X_2}{\partial F_2} - C_2 - f_2 = 0$$

So, the choice of optimal price and frequency of operator 2 is dependent on the chosen price and frequency of operator 1:

$$(A5) p_2 = p_2(p_1, F_1)$$

$$(A6) F_2 = F_2(p_1, F_1)$$

Operator 1, the leader; maximises:

$$(A7) \pi_1 = p_1 X_1[p_1, p_2[p_1, F]\ F_1,\ F_2[p_1, F]] - F_1 C_1 - c_{L1} X_1[] - f_1 F_1$$

The first-order conditions with respect to price and frequency are:

$$(A8) \frac{\partial \pi_1}{\partial p_1} = X_1 + (p_1 - c_{L1})(\frac{\partial X_1}{\partial p_1} + \frac{\partial X_1}{\partial p_2}\frac{\partial p_2}{\partial p_1} + \frac{\partial X_1}{\partial F_2}\frac{\partial F_2}{\partial p_1}) = 0$$

$$(A9)\frac{\partial \pi_1}{\partial F_1} = (\mathrm{p}_1 - \mathrm{c}_{L1})(\frac{\partial \mathrm{X}_1}{\partial F_1} + \frac{\partial \mathrm{X}_1}{\partial \mathrm{p}_2}\frac{\partial \mathrm{p}_2}{\partial F_1} + \frac{\partial \mathrm{X}_1}{\partial \mathrm{F}_2}\frac{\partial \mathrm{F}_2}{\partial F_1}) = 0$$

Differentiation of (A2) yields:

(A10)

$$\left[\frac{\partial \mathrm{X}_2}{\partial \mathrm{p}_2} + (\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{p}_2{}^2}\right]\mathrm{dp}_2 +$$
$$+\left[\frac{\partial \mathrm{X}_2}{\partial \mathrm{F}_2} + (\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{F}_2 \partial \mathrm{p}_2}\right]\mathrm{dF}_2 +$$
$$+\left[\frac{\partial \mathrm{X}_2}{\partial \mathrm{p}_1} + (\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{p}_1 \partial \mathrm{p}_2}\right]\mathrm{dp}_1 +$$
$$+\left[\frac{\partial \mathrm{X}_2}{\partial \mathrm{F}_1} + (\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{F}_1 \partial \mathrm{p}_2}\right]\mathrm{dF}_1 \equiv$$
$$\equiv \mathrm{adp}_2 + \mathrm{bdF}_2 + \mathrm{cdp}_1 + \mathrm{edF}_1 = 0$$
$$\left[\frac{\partial \mathrm{X}_2}{\partial \mathrm{F}_2} + (\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{p}_2 \partial \mathrm{F}_2}\right]\mathrm{dp}_2 +$$
$$+\left[(\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{F}_2{}^2}\right]\mathrm{dF}_2 +$$
$$+\left[(\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{p}_1 \partial \mathrm{F}_2}\right]\mathrm{dp}_1 +$$
$$+\left[(\mathrm{p}_2 - \mathrm{c}_{L2})\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{F}_1 \partial \mathrm{F}_2}\right]\mathrm{dF}_1 \equiv$$
$$\equiv \mathrm{fdp}_2 + \mathrm{gdF}_2 + \mathrm{hdp}_1 + \mathrm{idF}_1 = 0$$

The equation system is rewritten as:

$$(A11)\begin{pmatrix} a & b \\ f & g \end{pmatrix}\begin{pmatrix} dp_2 \\ dF_2 \end{pmatrix} = \begin{pmatrix} -cdp_1 - edF_1 \\ -hdp_1 - idF_1 \end{pmatrix}$$

$$(A12) dp_2 = \frac{\begin{vmatrix} -cdp_1 - edF_1 & b \\ -hdp_1 - idF_1 & g \end{vmatrix}}{ag - bf}$$

$$(A13)\frac{\partial p_2}{\partial p_1} = \frac{hb - cg}{ag - bf} \equiv \frac{\begin{vmatrix} b & c \\ g & h \end{vmatrix}}{\begin{vmatrix} a & b \\ f & g \end{vmatrix}}$$

$$(A14) dF_2 = \frac{\begin{vmatrix} a & -cdp_1 - edF_1 \\ f & -hdp_1 - idF_1 \end{vmatrix}}{ag - bf}$$

$$(A15)\frac{\partial F_2}{\partial p_1} = \frac{cf - ah}{ag - bf} \equiv \frac{\begin{vmatrix} c & a \\ h & f \end{vmatrix}}{\begin{vmatrix} a & b \\ f & g \end{vmatrix}}$$

$$(A16)\frac{\partial F_2}{\partial F_1} = \frac{ef - ia}{ag - bf} \equiv \frac{\begin{vmatrix} e & a \\ i & f \end{vmatrix}}{\begin{vmatrix} a & b \\ f & g \end{vmatrix}}$$

Total demand is assumed constant and normalised to one (1). The probability of choosing alternative 1, Pr(1), is the equal to demand of alternative 1, X_1. See expression (19). We distinguish between two cases that give rise to two different expressions for demand of alternative 2. Demand of alternative 1 is then simply 1 minus demand of alternative 2. In the expression below H_i is the interval of alternative i and x_i is the time till departure of alternative i.

Travel time cost except wait time of alternative i, R_i, is:

$$(A17) R_i = p_i + t_i$$

We select the alternative with minimum R_i and denote it R_1. We denote with k the difference between travel time cost except wait time of the other alternative, 2, and the least cost alternative, 1:

$$(A18) k = R_2 - R_1 \equiv p_2 + t_2 - p_1 - t_1$$

We express density of services in terms of frequency instead of headway, so that:

$$(A19)\frac{1}{F_i} = H_i$$

Case I is:

$$(A20) k \geq \frac{1}{F_1} - \frac{1}{F_2}$$

Case II is:

$$(A21) k < \frac{1}{F_1} - \frac{1}{F_2}$$

With only two alternatives taken into account there is an exact polynomial expression for the integral expression (19).

For case I demand of alternatives 2 and 1 are:

$$(A22) X_2 = \frac{1/F_1}{2/F_2}(1 - \frac{k}{1/F_1})^2 \equiv \frac{F_2}{2F_1} + \frac{1}{2}k^2 F_1 F_2 - kF_2$$

$$(A23) X_1 = 1 - X_2 \equiv 1 - (\frac{F_2}{2F_1} + \frac{1}{2}k^2 F_1 F_2 - kF_2)$$

For case II demand of alternatives 2 and 1 are:

$$(A24) X_2 = 1 - \frac{k}{1/F_1} - \frac{1/F_2}{2/F_1} \equiv 1 - kF_1 - \frac{F_1}{2F_2}$$

$$(A25) X_1 = 1 - X_2 \equiv kF_1 + \frac{F_1}{2F_2}$$

For case I we have the following first and second order conditions.

(A26)

$$\frac{\partial \mathrm{X}_2}{\partial \mathrm{p}_2} = \frac{\partial \mathrm{X}_1}{\partial \mathrm{p}_1} = -\mathrm{F}_2 + \mathrm{kF}_1\mathrm{F}_2 \qquad \frac{\partial \mathrm{X}_2}{\partial \mathrm{p}_1} = \frac{\partial \mathrm{X}_1}{\partial \mathrm{p}_2} = \mathrm{F}_2 - \mathrm{kF}_1\mathrm{F}_2$$

$$\frac{\partial X_2}{\partial F_2} = \frac{1}{2F_1} + \frac{k^2 F_1}{2} - k \qquad \frac{\partial X_2}{\partial F_1} = -\frac{F_2}{2F_1^{\ 2}} + \frac{k^2 F_2}{2}$$

$$\frac{\partial X_1}{\partial F_1} = \frac{F_2}{2F_1^{\ 2}} - \frac{k^2 F_2}{2} \qquad \frac{\partial X_1}{\partial F_2} = -\frac{1}{2F_1} - \frac{k^2 F_1}{2} + k$$

$$\frac{\partial^2 \mathrm{X}_2}{\partial \mathrm{p}_2^{\ 2}} = \mathrm{F}_1\mathrm{F}_2 \qquad \frac{\partial^2 X_2}{\partial p_2 \partial F_2} = kF_1 - 1$$

$$\frac{\partial^2 X_2}{\partial p_1 \partial F_2} = -kF_1 + 1 \qquad \frac{\partial^2 X_2}{\partial F_2^{\ 2}} = 0$$

$$\frac{\partial^2 X_2}{\partial p_1 \partial p_2} = -F_1 F \qquad \frac{\partial^2 X_2}{\partial F_1 \partial F_2} = -\frac{1}{2F_1^{\ 2}} + \frac{k^2}{2}$$

$$\frac{\partial^2 X_2}{\partial p_2 \partial F_1} = kF_2$$

For case II we have the following first and second order conditions.

(A27)

$$\frac{\partial X_2}{\partial p_2} = \frac{\partial X_1}{\partial p_1} = -F_1 \qquad \frac{\partial X_2}{\partial p_1} = \frac{\partial X_1}{\partial p_2} = F_1$$

$$\frac{\partial X_2}{\partial F_2} = \frac{F_1}{2F_2{}^2} \qquad \frac{\partial X_2}{\partial F_1} = -k - \frac{1}{2F_2}$$

$$\frac{\partial X_1}{\partial F_2} = -\frac{-F_1}{2F_2{}^2} \qquad \frac{\partial X_1}{\partial F_1} = k + \frac{1}{2F_2}$$

$$\frac{\partial^2 X_2}{\partial p_1 \partial p_2} = 0 \qquad \frac{\partial^2 X_2}{\partial F_1 \partial p_2} = -1$$

$$\frac{\partial^2 X_2}{\partial p_2{}^2} = 0 \qquad \frac{\partial^2 X_2}{\partial p_2 \partial F_2} = 0$$

$$\frac{\partial^2 X_2}{\partial p_1 \partial F_2} = 0 \qquad \frac{\partial^2 X_2}{\partial F_2{}^2} = \frac{-F_1}{F_2{}^3}$$

$$\frac{\partial^2 X_2}{\partial F_1 \partial F_2} = \frac{1}{2F_2{}^2}$$

By applying the derivates above, we can express the elements of the cross derivatives for case 1 as follows:

(A28)

$$a = F_2(kF_1 - 1) + (p_2 - c_{L\,2})F_1F_2$$

$$b = f = \frac{1}{2F_1} + \frac{k^2F_1}{2} - k + (p_2 - c_{L\,2})(kF_1 - 1)$$

$$c = F_2(1 - kF_1) + (p_2 - c_{L\,2})(-F_1F_2)$$

$$e = -\frac{F_2}{2F_1^2} + \frac{k^2F_2}{2} + (p_2 - c_{L\,2})kF_2$$

$$g = 0$$

$$h = (p_2 - c_{L\,2})(1 - kF_1)$$

$$i = (p_2 - c_{L\,2})(-\frac{1}{2F_1^2} + \frac{k^2}{2})$$

The cross derivatives of case I are then:

$$(A29)\ \frac{\partial p_2}{\partial p_1} = \frac{-h}{f} \equiv \frac{-(p_2 - c_{L\,2})(1 - kF_1)}{\frac{1}{2F_1} + \frac{k^2F_1}{2} - k + (p_2 - c_{L\,2})(kF_1 - 1)}$$

$$(A30)\frac{\partial p_2}{\partial F_1}=\frac{-i}{f}\equiv\frac{-(\mathrm{p}_2-\mathrm{c}_{L2})(-\frac{1}{2F_1^2}+\frac{k^2}{2})}{\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1)}$$

$$(A31)\frac{\partial F_2}{\partial p_1}=\frac{ah-cf}{bf}\equiv\frac{(F_2(kF_1-1)+(\mathrm{p}_2-\mathrm{c}_{L2})F_1F_2)(\mathrm{p}_2-\mathrm{c}_{L2})(1-kF_1)-(F_2(1-kF_1)+(\mathrm{p}_2-\mathrm{c}_{L2})(-F_1F_2))(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))}{(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))^2}$$

$$(A32)\frac{\partial F_2}{\partial F_1}=\frac{ai-ef}{bf}\equiv\frac{(F_2(kF_1-1)+(\mathrm{p}_2-\mathrm{c}_{L2})F_1F_2)(\mathrm{p}_2-\mathrm{c}_{L2})(-\frac{1}{2F_1^2}+\frac{k^2}{2})-(-\frac{F_2}{2F_1^2}+\frac{k^2F_2}{2}+(\mathrm{p}_2-\mathrm{c}_{L2})kF_2)(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))}{(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))^2}$$

The first-order conditions with respect to price and frequency of the leader are then expressed as:

$$(A33)0=\mathrm{X}_1+(\mathrm{p}_1-\mathrm{c}_{L1})(F_2(kF_1-1)-F_2(kF_1-1)\frac{-(\mathrm{p}_2-\mathrm{c}_{L2})(1-kF_1)}{\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1)}+$$
$$+(-\frac{1}{2F_1}-\frac{k^2F_1}{2}+k)\frac{(F_2(kF_1-1)+(\mathrm{p}_2-\mathrm{c}_{L2})F_1F_2)(\mathrm{p}_2-\mathrm{c}_{L2})(1-kF_1)-(F_2(1-kF_1)+(\mathrm{p}_2-\mathrm{c}_{L2})(-F_1F_2))(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))}{(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))^2}$$

$$(A34)0=(p_1-c_{L1})(\frac{F_2}{2F_1^2}-\frac{k^2F_2}{2}+(\mathrm{F}_2-\mathrm{kF}_1\mathrm{F}_2)\frac{-(\mathrm{p}_2-\mathrm{c}_{L2})(-\frac{1}{2F_1^2}+\frac{k^2}{2})}{\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1)}+$$
$$+(-\frac{1}{2F_1}-\frac{k^2F_1}{2}+k)\frac{(F_2(kF_1-1)+(\mathrm{p}_2-\mathrm{c}_{L2})F_1F_2)(\mathrm{p}_2-\mathrm{c}_{L2})(-\frac{1}{2F_1^2}+\frac{k^2}{2})-(-\frac{F_2}{2F_1^2}+\frac{k^2F_2}{2}+(\mathrm{p}_2-\mathrm{c}_{L2})kF_2)(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))}{(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k+(\mathrm{p}_2-\mathrm{c}_{L2})(kF_1-1))^2})$$

The first-order conditions with respect to price and frequency of the follower are:

$$(A35)0=X_2+(\mathrm{p}_2-c_{L2})(-\mathrm{F}_2+\mathrm{kF}_1\mathrm{F}_2)$$

$$(A36)0=(\mathrm{p}_2-c_{L2})(\frac{1}{2F_1}+\frac{k^2F_1}{2}-k)-C_2-f_2$$

where we note that:

$$(A37)k=R_2-R_1\equiv p_2+t_2-p_1-t_1$$

This problem has no analytical solution. Furthermore, the two cases and the boundaries for acceptance of one or the other alternative give rise to six regions where optima are functions of how prices, ride times and frequencies relate between the two operators. Numerical calculation is probably the only way forward, but that issue has to stay pending.